An Investigation into Greek Computations: Malliavin Estimators and Deep Learning

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Introduction

An *option* is a financial contract that allows one to either buy or sell (buy for call, sell for put) a certain quantity of an asset at an agreed exercise price K, called the *strike* and an agreed time T, called the *maturity date*. If the buyer of the option exercises their right granted by the option, then the writer has the obligation to purchase or sell the asset at the strike. In exchange for being given this right, the buyer of the option must pay a fee x, called the *premium*. Seeing how the premium of an option changes with respect to certain parameters, such as the stock price or volatility, is of particular importance to practitioners of the financial industry, as one can then gauge the feasibility and fairness of an option.

Greeks and Types of Options

The most simple type of option are the class of *European* options, which only allow the buyer of the option to exercise at the established time T. We denote by S_t the observed price of our asset at time $0 \le t \le T$, and by V_T the option price. The value of a European call and put option at maturity are $(S_T - K)^+ = \max(0, S_T - K)$, and $(K - S_T)^+ = \max(0, K - S_T)$ respectively. Often, these option contracts are settled with cash instead of physical delivery at maturity, meaning that in general we only need to specify a *payoff* function $\phi(S_T)$, that specifies the amount of money the writer owes the holder at T. As above, the payoff function of a European call will be

$$\phi(S_T) = (S_T - K)^+.$$

This payoff structure is called *vanilla* because it is not complicated. More generally, ϕ can be any function that can be explicitly computed on day T. In mathematical finance, we often want to estimate $V_0 = \mathbb{E}_{\mathbb{Q}}[\phi]$. The problem is that as market conditions evolve, such as S_t, the current price of the asset at time t, r, the interest rate, and σ, the volatility. These factors will change V_0; this estimate is "sensitive" to these certain parameters. The sensitivities of V_0 we call the *Greeks*.

Name	Sensitivity	Formula
Delta (Δ)	Price (S)	$\partial V/\partial S$
Vega $(\mathcal{V})$	Volatility (σ)	$\partial V/\partial \sigma$
Theta (Θ)	Time (τ)	$-\partial V/\partial \tau$
Rho (ρ)	Interest Rate (r)	$\partial V/\partial r$
Gamma (Γ)	Delta (Δ)	$\partial \Delta/\partial S$

The next type of options are the *Asian options*. These options are apart of the class of "path-dependent" options, meaning that the payoff depends on the movement of the stock itself. Here we choose to take the average continuously, so that the parameters we input into our payoff function will be

$$\phi(\bar{S}_T) = \phi\left(\int_0^T S_t\, dt\right).$$

This means we can write the value of an Asian call and put at time T as $(\frac{1}{T}\bar{S}_T - K)^+$ and $(K - \frac{1}{T}\bar{S}_T)^+$ respectively.

A *binary* or *digital* option is an exotic option in which the payoff is either some fixed amount or nothing at all if a criterion is met. In essence, these are a specific case of a larger set of payoff functions, as their payoff functions are dependent on the structure of S_T. This means that we can have European digital options or Asian digital options, for example. These have mathematical formulation

$$\phi(S_T) = \mathbb{1}_{\{S_T > K\}}; \quad \phi(\bar{S}_T) = \mathbb{1}_{\{\bar{S}_T > K\}}.$$

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References

[1] D. Nualart. *The Malliavin Calculus and Applications*. Springer, 1995.

Methods

There are already many methods for numerically approximating the Greeks: the finite difference method, the likelihood method, and the pathwise derivative method. [1] introduces a way to use Malliavin calculus, the "stochastic calculus of variations" developed by Paul Malliavin in the late 1970s. The beauty of Malliavin calculus is that is gives us a way to use integration by parts in the world of stochastic processes, which are by construction non-differentiable.

How do we rigorously define the derivative operator DF? We must have a square integrable random variable $F : \Omega \to \mathbb{R}$, where the derivative is taken w.r.t. the parameter $\omega \in \Omega$.

Definition 1 (Malliavin Derivative)**.** Consider the set of smooth random variables $\mathcal{S}$:

$$\mathcal{S} = \{F = f(W(h_1), \dots, W(h_n)),\ f \in \mathcal{C}_p^\infty(\mathbb{R}^n),\ h_i \in H,\ n \ge 1\},$$

where $\mathcal{C}_p^\infty(\mathbb{R}^n)$ is the set of functions $f : \mathbb{R}^n \to \mathbb{R}$ that are $\mathcal{C}^\infty$ and such that f and all its partial derivatives have polynomial growth. If $F \in \mathcal{S}$, we define the *derivative* of F, denoted DF, as the $H-$valued random variable:

$$DF = \sum_{i=1}^n \partial_i f(W(h_1), \dots, W(h_n)) h_i.$$

We denote the domain of D in $L^p(\Omega)$ by the Sobolev space $\mathbb{D}^{1,p}$ for any $p \ge 1$, which means that $\mathbb{D}^{1,p}$ is the closure of $\mathcal{S}$ w.r.t. the norm

$$\|F\|_{1,p} = (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathcal{H}}^p])^{1/p}.$$

We can interpret $\mathbb{D}^{1,p}$ as an infinite dimensional weighted Sobolev space, and for $p = 2$, we have a Hilbert space with scalar product

$$\langle F, G \rangle_{1,2} = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG \rangle_{\mathcal{H}}].$$

The adjoint of the Malliavin derivative gives us an analogous integral operator:

Definition 2 (Skorokhod Integral)**.** Let δ be an unbounded operator on $L^2(\Omega; \mathcal{H})$ with values in $L^2(\Omega)$ such that the domain of δ, $\mathrm{Dom}(\delta)$, is the set of $\mathcal{H}-$valued square integrable random variables $u \in L^2(\Omega; \mathcal{H})$ satisfying

$$|\mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}]| = \mathbb{E}\left[\int_0^\infty D_t F u(t)\, dt\right] \le K(u) \|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where $K(u)$ is some constant depending on u but independent of F. If u belongs to $\mathrm{Dom}(\delta)$, then the Skorokhod integral operator $\delta(u)$ is defined as the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}], \quad \forall F \in \mathbb{D}^{1,2}.$$

The elements of $\mathrm{Dom}(\delta) \subset L^2([0,T] \times \Omega)$ are square integrable processes, and

$$\delta(u) = \int_0^T u(t) \delta W_t,$$

which is known as the *Skorokhod integral* of the process u.

Theorem 1 (Integration by Parts)**.** Let F, G be two random variables such that $F \in \mathbb{D}^{1,2}$. Let u be an $H-$valued random variable such that $\langle DF, u \rangle_H \neq 0$ almost surely and $Gu(\langle DF, u \rangle_H)^{-1} \in \mathrm{Dom}\,\delta$. Then for any function $f \in \mathcal{C}^1$ with bounded derivatives, we have that

$$\mathbb{E}[f'(F)G] = \mathbb{E}[f(F)H(F,G)],$$

where $H(F,G) = \delta(G(\langle DF, u \rangle_H)^{-1}$.

For this project, we extend the work of [1]. In particular, we use the Integration by Parts formula to develop the following forms for Γ and $\mathcal{V}$ of an Asian option with continuous averaging:

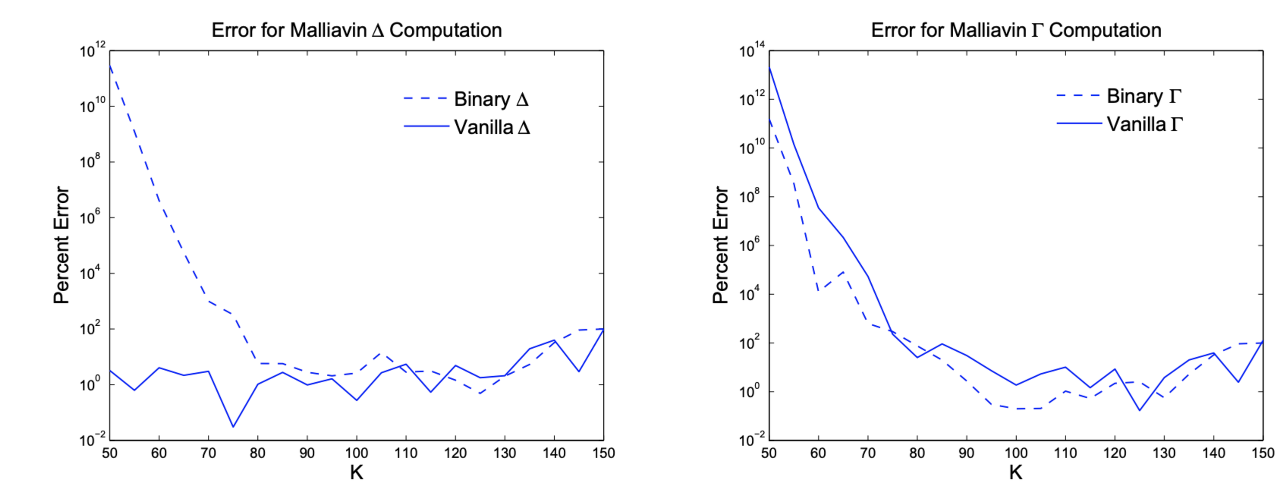
$$\Gamma_0 = \frac{4e^{-rT}}{\sigma^3 S_0^2} \mathbb{E}\left[\phi(\bar{S}_T)\left(\frac{(S_T - S_0)^2 - (S_T - S_0) r \bar{S}_T}{\sigma \bar{S}_T^2} - \frac{\sigma S_0}{\bar{S}_T}\right)\right] - \frac{2r}{\sigma^2 S_0}\Delta.$$

$$\mathcal{V}_0 = e^{-rT}\mathbb{E}\left[\phi(\bar{S}_T)\left(\frac{\int_0^T \left(\int_0^T S_t W_t\, dt\right) dW_s}{\sigma \int_0^T t S_t\, dt} + \frac{1}{S_0}\int_0^T S_t W_t\, dt \frac{\int_0^T t^2 S_t\, dt}{\left(\int_0^T t S_t\, dt\right)^2} - W_T\right)\right].$$

The new form for Γ in terms of Δ eases computational costs significantly compared to older models.

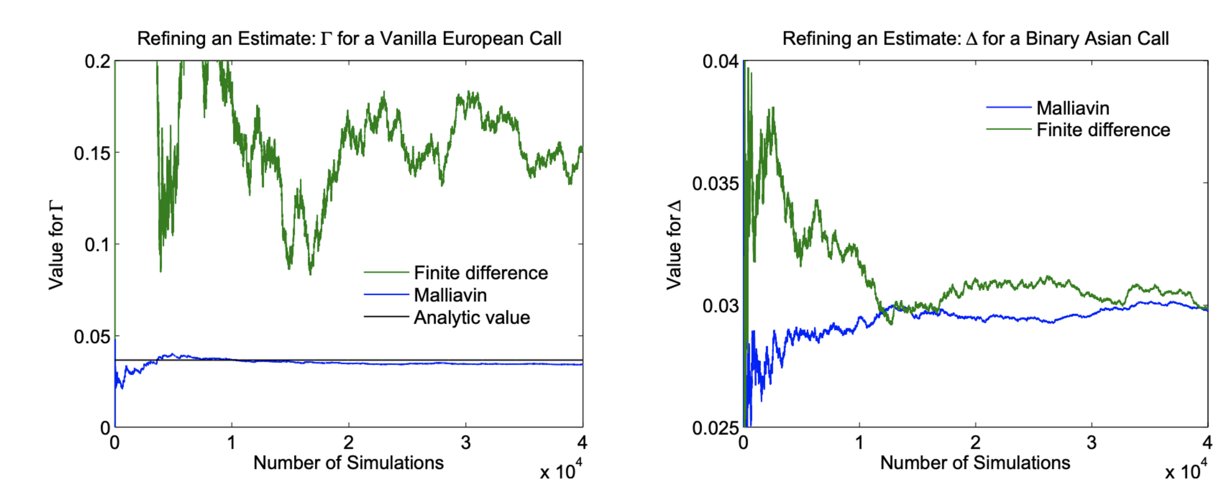
Numerical Results

To simulate the Greeks and compare our Malliavin estimator to currently existing estimators, we assume S_t follows a Geometric Brownian motion with conditions $S_0 = 100$, $r = 0.05$, $\sigma = 0.10$, and maturity $T = 1$, and time discretized to a small number of paths.



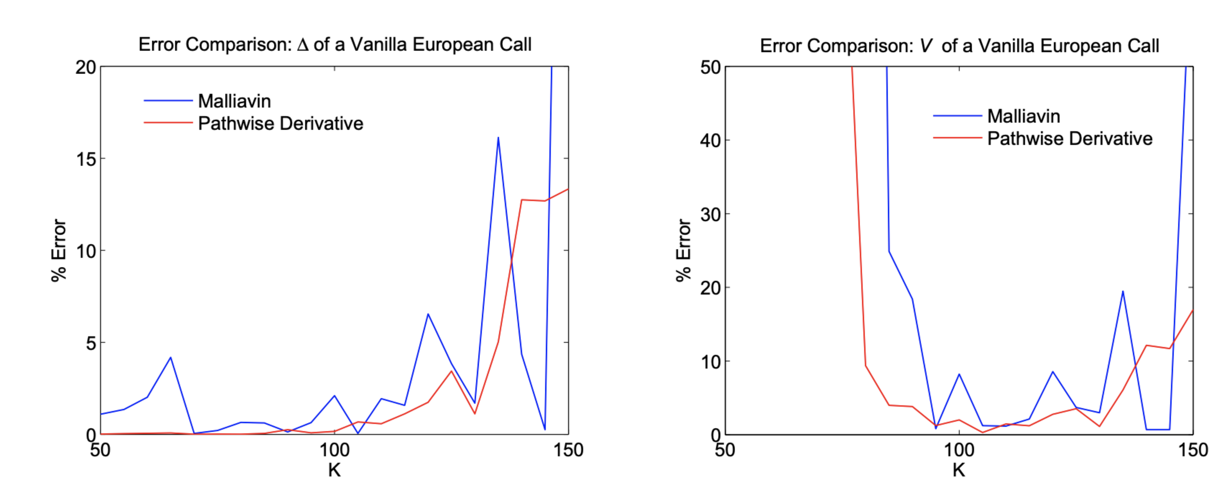
(a) Vanilla (b) Binary

Fig. 1: Comparing errors for various strikes on European and binary options



(a) Γ for a vanilla European call, $K = 110$. (b) Δ for a binary Asian call, $K = 110$.

Fig. 2: Refinement of Malliavin and finite difference estimators



(a) Δ for a vanilla European call. (b) $\mathcal{V}$ for a vanilla European call.

Fig. 3: Comparison of errors for Malliavin and pathwise derivative estimators

The following can be concluded:

- The Malliavin estimator is more accurate than the finite difference estimator by at least one order of magnitude.
- The Malliavin estimator is better than the pathwise derivative estimator because the latter cannot be used to compute nonvanilla options or greeks involving the second derivative.
- The pathwise derivative estimator slightly outperforms the Malliavin estimators when it is applicable.

Future Work: Deep Learning

Another approach to the numerical estimation problem is one involving that of deep learning.